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Problem Set I

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1. (a) Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a one-one, continuous function. Show that f is homeomorphism onto the range of f.
   (b) Does the result hold if $\mathbf{R}$ is replaced by $\mathbf{R}^n$ in part (a) ?
   (c) What happens if $\mathbf{R}$ is replaced by $\mathbf{S}^n$ in part (a) ?
   (d) Prove that a real valued bijective continuous function on a compact space is a homeomorphism.
   (e) Generalize (d) by replacing the codomain by a suitable topological space.

2. (a) Show that an open set in $\mathbf{R}$ can be expressed as a countable union of open intervals.
   (b) What if $\mathbf{R}$ is replaced by $\mathbf{R}^n$ and intervals by open cubes ?

3. Is $\mathbf{R}^2 \setminus Q \times Q$ connected ?

4. Prove that every convex subset of $\mathbf{R}^n$ is connected.

5. Let A be an open subset of $\mathbf{R}$ and f be real valued function on $\mathbf{R}$ such that $f'(x) \neq 0, \forall\ x \in A$.
   (a) Is f one-one ?
   (b) Is $f(A)$ open ?
   (c) What if $\mathbf{R}$ is replaced by $\mathbf{R}^n,\ n \geq 2$?

6. (a) If f is a real valued continuous function on an open interval, then show that f has a primitive.
   (b) If $f = (f_1, f_2)$ is continuous function on an open set in $\mathbf{R}^2$, then is there a real valued smooth function F such that

$$\frac{\partial F}{\partial x_1} = f_1 \text{ and } \frac{\partial F}{\partial x_2} = f_2$$